A Quick and Efficient Method for Determining the Scattering Matrix of Lossless Microwave Circuits

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***Abstract*—The measurements of scattering coefficients of an n-port lossless circuit with a VNA may be tedious because the experimental results may be incompatible with each other and do not verify the conditions which relate the coefficients of the unitary matrix. In this letter, we show that it is efficient to fuse the data by calculating, in one step, the unitary part of the primary experimental scattering matrix.**

***Index Terms*—Coherency scattering coefficients, fusion of VNA data, measurements of scattering coefficients, VNA calibration errors.**

I. Introduction

The first step in studying microwave n port circuits is generally determining the scattering matrix (1). The measurements of scattering coefficients are performed by connecting a network analyzer to any ports i and j and loading the other ports with adapted load circuits. The measurements are done again successively with all pairs of different i and j ports. Each experiment gives three complex scattering coefficients.

The construction of the $[S]$ matrix by collecting the data is sometimes tedious and may pose problems. There are errors in measurement, mainly due to the inaccuracy in the calibration procedure. Some measured values may be incoherent with others because the adapted load circuits which are used are not perfect and thus have small reflexion coefficients. Difficulties may arise specially for lossless circuits, such as those used at high power levels in industrial techniques (2). The experimental data may be incompatible with each other since the matrix $[S]$, which should be symmetric and unitary when the circuit studied is lossless, does not verify unitary equations. A matrix $[S]$ is unitary when its product with its transposed complex conjugate matrix $[\tilde{S}^*]$ yields the identity matrix. Generally, the conditions of unitarity reduce the number of independent algebraic unknowns (from the complex coefficients S_{ij}) by a factor of two.

II. Processing Method

The fusion of data may be done by applying a least squares method to determine the unknown coefficients. This is nevertheless tedious, because it leads to optimizing by a least squares procedure $n(n+1)/2$ algebraic parameters. In the following note, we propose to check the unitary conditions of the matrix $[S]$ by calculating a unitary matrix $[\Sigma]$, which is numerically near $[S]$, in one step. The method uses the theorem we recall in the appendix (3). The product $[S] \times \left[\tilde{S}^*\right] (= [H]^2)$ is a matrix which is approximately the identity matrix. For the low loss n port, it would be exactly $[I]$ if the measurements were perfect. Its eigenvalues (α_i) are close to 1. Let $[V_i]$ and $[\tilde{V}_i^*]$ be the matrices constructed with the eigenvectors which diagonalize $[S] \times \left[\tilde{S}^*\right] = [V_i] \times [\alpha_i] \times \left[\tilde{V}_i^*\right]$.

The diagonal matrices $[\ \alpha_i]$, $\left[\sqrt{\alpha_i}\right]$ and $[1/\sqrt{\alpha_i}]$ are also approximately the identity matrices and

$$[\Sigma] = [V_i] \times \left[\frac{1}{\sqrt{\alpha_i}}\right] \times \left[\tilde{V}_i^*\right] \times [S]$$

is unitary as the theorem in the appendix states.

The matrix $[V_i] \times \left([I] - \left[1/\sqrt{\alpha_i}\right]\right) \times \left[\tilde{V}_i^*\right]$ appears to be a factor which corrects $[S]$ so it becomes unitary. In cases when measurement anomalies due to defects such as a misalignment, misconnection of waveguide, or a local resonance exist, $[\Sigma]$ corrects the experimental determination of the circuit scattering matrix. It is a better correction than a conventional least squares optimization, the use of which is not justified when the errors, to be smoothed, have no statistical distribution. The optimization procedure never gives an exact unitary result and spreads the anomaly errors over all terms of the matrix.

Since the matrix $[\Sigma]$, is rigorously unitary, it is worth calculating the impedance and the admittance matrices by using the well known formulae (4). Their coefficients are purely imaginary numbers. It is possible, then, to check the coherency with direct measurement of some of them, when measurements are done with a short circuit, or an open circuit, loading some ports.

As illustrative examples, let us consider two cases.

a) With a HP 8719C vector network analyzer, we obtained for a thin, metallic obstacle in a WR340 waveguide, at 2450 MHz $S_{11}^* = 0.233\,\underline{|-139^\circ}$; $S_{12}^* = 0.973\,\underline{|-15^\circ}$; $S_{21}^* = 0.915\,\underline{|-13^\circ 57}$ and $S_{22}^* = 0.235\,\underline{|-141^\circ}$. S_{11}^* differs from S_{22}^* and $S_{12}^* \neq S_{21}^*$. The $[S]$ matrix is not symmetric and not unitary. The calculation of $[\Sigma]$ gives $\Sigma_{11}^* = \Sigma_{22}^* = 0.203\,\underline{|-120^\circ}$ and $\Sigma_{12}^* = \Sigma_{21}^* = 0.979\,\underline{|-30^\circ}$. The data are now coherent within ± 3 mU in modulus and $\pm 2^\circ$ in phase. Although this example is trivial, because the unitary conditions could be satisfied with hand calculation, it shows the potentiality of doing the verification automatically.

b) We measured the $[S]$ matrix of a home made tee: $S_{11}^* = 0.420\,\underline{|-18^\circ}$; $S_{12}^* = 0.049\,\underline{|-140^\circ}$; $S_{13}^* = 0.670\,\underline{|56^\circ}$; $S_{14}^* = 0.670\,\underline{|56^\circ}$; $S_{22}^* = 0.630\,\underline{|104^\circ}$; $S_{23}^* = 0.548\,\underline{|-120^\circ}$; $S_{24}^* = 0.540\,\underline{|-111^\circ}$; $S_{33}^* = 0.49\,\underline{|-116^\circ}$; S_{34}^* = not measured and

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$S_{44}^* = 0.490\,\underline{|-116^\circ}$. The unitary conditions are not fulfilled, although the trace of $[S]$ is $4.09 + 0.00i$. The largest coefficient of $[S] \times \left[\tilde{S}^*\right]$ is 0,3 in modulus. The first calculation of $[\Sigma]$ detects an error in phase measurement of S_{22}^* and S_{24}^*. The second calculation of $[\Sigma]$ yields: $\Sigma_{11} = 0.409\,\underline{|-26^\circ}$, $\Sigma_{12} = 0.08\,\underline{|-78^\circ}$, $\Sigma_{13} = 0.623\,\underline{|58.2^\circ}$, $\Sigma_{14} = 0.667\,\underline{|55.9^\circ}$, $\Sigma_{21} = 0.084\,\underline{|178^\circ}$, $\Sigma_{22} = 0.378\,\underline{|-37^\circ}$, $\Sigma_{23} = 0.639\,\underline{|-119^\circ}$, $\Sigma_{24} = 0.672\,\underline{|59^\circ}$, $\Sigma_{31} = 0.670\,\underline{|59.1^\circ}$, $\Sigma_{33} = 0.631\,\underline{|-119.7^\circ}$, $\Sigma_{33} = 0.375\,\underline{|-37^\circ}$, $\Sigma_{34} = 0.705\,\underline{|-172^\circ}$, $\Sigma_{41} = 0.666\,\underline{|56^\circ}\Sigma_{42} = 0.666\,\underline{|56^\circ}\Sigma_{43} = 0.080\,\underline{|-170^\circ}\Sigma_{44} = 0.411\,\underline{|-27^\circ}$. The directivity (for the non-adapted tee) is estimated to be -11 db and the symmetry between branches three and four is within $\pm 3\%$. The trace of $[S] \times \left[\tilde{S}^*\right]$ is now 4,0001 and the sum of the moduli of all $[S] \times \left[\tilde{S}^*\right]$ coefficients is 4,003. A factor of 10 improvement in $[S]$ determination accuracy is obtained (after adaptation of the hybrid tee, so its directivity is reduced to −41 db.)

III. Conclusion

The use of the decomposition theorem of any non singular matrix into two hermitian and unitary matrices simplifies the fusion of data in scattering matrix determination of a no loss circuit.

Appendix

Any non singular matrix $[A]$ can be decomposed (3) into the product of a hermitian matrix by an unitary matrix.

$$[A] = [H] \times [U]$$

The eigenvalues α_i and the eigenvectors V_i of the matrix $[A] \times \left\lfloor \tilde{A}^* \right\rfloor$ are first calculated so that

$$[A] \times \left[\tilde{A}^*\right] = [V_i] \times [\alpha_i] \times \left[\tilde{V}_i^*\right]$$

then

$$[H] = [V_i] \times \lfloor \sqrt{\alpha_i} \rfloor \times \left[\tilde{V}_i^*\right]$$

and

$$[U] = [H]^{-1} \times [A]$$

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